

# On Generalized West and Stampfli Decomposition of Operators

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## Abstract

In this paper, we characterize meromorphic operators in terms of B-Fredholm operators and operators of topological uniform descent. Using those characterizations, we recover several new results and earlier results on meromorphic operators established in [9], [12], [17], [20] and [27]. Then, we define operators with purely B-discrete spectrum and we prove that they are exactly the operators with a meromorphic resolvent. Moreover, we define operators with generalized Riesz or generalized meromorphic resolvent and we prove that an operator  $T$  with a generalized meromorphic resolvent has a generalized meromorphic West-Stampfli decomposition  $T = M + Q + D$ , where  $M$  is meromorphic,  $Q$  quasinilpotent such that the commutator  $[M, Q]$  is quasinilpotent and  $D$  is a diagonal operator whose spectrum is the B-Fredholm spectrum of  $T$ . Similar decomposition is obtained for operators having generalized Riesz resolvent.

## 1 Introduction

Let  $\mathcal{C}(X)$  be the set of all linear closed operators defined from a Banach space  $X$  to  $X$  and  $L(X)$  be the Banach algebra of all bounded linear operators defined from  $X$  to  $X$ . We write  $D(T)$ ,  $N(T)$  and  $R(T)$  for the domain, nullspace and range of an operator  $T \in \mathcal{C}(X)$ . An operator  $T \in \mathcal{C}(X)$  is called a *Fredholm* operator [22] if both the nullity of  $T$ ,  $n(T) = \dim N(T)$ , and the defect of  $T$ ,  $d(T) = \operatorname{codim} R(T)$ , are finite. The index  $i(T)$  of a Fredholm operator  $T$  is defined by  $i(T) = n(T) - d(T)$ . It is well known that if  $T$  is a Fredholm operator, then  $R(T)$  is closed.

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The class of bounded linear B-Fredholm operators, which is a natural extension of the class of Fredholm operators, was introduced in [2], and the class of unbounded linear closed B-Fredholm operators acting on a Banach space was studied in [6].

Recall [8] that a linear bounded operator is called a meromorphic operator if  $\lambda = 0$  is the only possible point of accumulation of  $\sigma(T)$  and every non-zero isolated point of  $\sigma(T)$  is a pole of the resolvent  $R_\mu(T) = (T - \mu I)^{-1}$ , defined on the resolvent set  $\rho(T)$  of  $T$ . If we also require that each non-zero eigenvalue of  $T$  have finite multiplicity, then  $T$  will be called a Riesz operator.

A first result linking bounded B-Fredholm operators to the class  $\mathfrak{M}$  of linear bounded meromorphic operators comes from the following theorem, established in [5, Theorem 2.11].

**THEOREM 1.1.** *Let  $T \in L(X)$ . Then  $T$  is a meromorphic operator if and only if  $\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a B-Fredholm operator}\} \subset \{0\}$ .*

As it will be seen in the second section, we extend Theorem 1.1 by showing that it holds also if we replace  $\sigma_{BF}(T) \subset \{0\}$  by the weaker hypothesis  $\sigma_{TUD}(T) \subset \{0\}$ , where  $\sigma_{TUD}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not an operator of topological uniform descent}\}$  is the topological uniform descent spectrum of  $T$ . This enable us to recover some earlier results on Riesz or meromorphic operators obtained in [9, Theorem 5], [17, Theorem 3] and [20, Theorem 4.2], and to obtain “polynomially perturbed” versions of recent results on polynomially Riesz or polynomially meromorphic operators obtained in [27, Theorem 2.13] and [12, Theorem 4.2]. Recall that the class of operators of topological uniform descent had been introduced in [14] and in [3] it was shown that it forms a regularity [3, Definition 1.2]. We will show also that Theorem 1.1 characterizes the class of unbounded meromorphic operators  $\mathfrak{M}(0, \infty)$  studied in [10].

For a closed operator  $T$  acting a Banach space, the Weyl spectrum  $\sigma_w(T)$  of  $T$ , is defined [22] as the set of all complex numbers  $\lambda$  such that  $T - \lambda I$  is not a Fredholm operator of index 0. The Weyl spectrum  $\sigma_w(T)$  of  $T$  is a subset of the spectrum  $\sigma(T)$  and its complement in  $\sigma(T)$  is called the discrete spectrum, that is  $\sigma_d(T) = \sigma(T) \setminus \sigma_w(T)$ .

Analogously, we define here the B-discrete spectrum for closed operators, as a natural extension of the discrete spectrum.

**DEFINITION 1.2.** Let  $T \in \mathcal{C}(X)$ . Then  $T$  is called a B-Weyl operator if it is a B-Fredholm operator of index 0. The B-Weyl spectrum  $\sigma_{BW}(T)$  of  $T$  is defined by  $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}$ . The B-discrete spectrum  $\sigma_{BD}(T)$  of  $T \in \mathcal{C}(X)$  is defined by  $\sigma_{BD}(T) = \sigma(T) \setminus \sigma_{BW}(T)$ .

It's clear that both of the B-essential spectrum and the B-Discrete spectrum of  $T$  are subsets of the spectrum of  $T$ .

DEFINITION 1.3. We will say that  $T$  has a purely B-discrete spectrum if  $\sigma(T) = \sigma_{\text{BD}}(T)$ , and we will say that  $T$  has a finite purely B-discrete spectrum if  $\sigma_{\text{BD}}(T)$  is also finite.

In the third section, we characterize closed operators with non-empty resolvent set, having a purely B-discrete spectrum, by showing that this is the case if and only if the operator considered has a meromorphic resolvent.

Moreover, using the Weyl's criterion, we show that if  $T$  is a self-adjoint operator acting on a Hilbert space, its B-discrete spectrum  $\sigma_{\text{BD}}(T)$  coincides with the isolated point of its spectrum, while the accumulation points of the spectrum  $\sigma(T)$  of  $T$  forms its B-Weyl spectrum  $\sigma_{\text{BW}}(T)$ . Thus a natural partition of the spectrum of a self-adjoint operator is reached via these two distinguished parts of the spectrum of  $T$  as  $\sigma(T) = \sigma_{\text{BD}}(T) \sqcup \sigma_{\text{BW}}(T) = \text{acc}(\sigma(T)) \sqcup \text{iso}(\sigma(T))$ . Here the symbols  $\sqcup$ ,  $\text{acc}$  and  $\text{iso}$  stands respectively for disjoint union, accumulation points and isolated points.

The discrete spectrum has important applications in the study of physical operators. However, the discrete spectrum does not remove from the spectrum poles of the resolvent which are of infinite rank, while the B-discrete spectrum does. An illustrating example of an operator with purely B-discrete spectrum, is given by the Schrödinger operator with a constant magnetic field  $B \neq 0$ . Its B-discrete spectrum coincides exactly with the set of its Landau levels, while its discrete spectrum is the empty set, (see Example 3.6).

In the fourth section, we consider operators having generalized Riesz or generalized meromorphic resolvent (Definition 4.1). Using results on meromorphic West-Stampfli decomposition obtained in [18], we prove that an operator  $T$  with a generalized meromorphic resolvent has a generalized meromorphic West-Stampfli decomposition  $T = M + Q + D$ , where  $M$  is a meromorphic operator,  $Q$  is quasinilpotent operator such that the commutator  $[M, Q]$  is quasinilpotent and  $D$  is a diagonal operator whose spectrum is exactly the B-Fredholm spectrum of  $T$ . Moreover, we extend a result of Koliha [18, Theorem 4.13], to the case of operators having a generalized Riesz resolvent by proving that if  $\sum_{n=1}^{\infty} |\nu_n| < \infty$  where  $(\nu_n)_n$  are all isolated eigenvalues of  $T$  each repeated according to its multiplicity, then  $T$  has a generalized West decomposition  $T = K + Q + D$ , where  $K$  is a compact operator,  $Q$  is a quasinilpotent operator such that the commutator  $[K, Q]$  is quasinilpotent and  $D$  is a diagonal operator whose spectrum is exactly the Browder spectrum  $\sigma_{\text{B}}(T)$  of  $T$ . Recall that a Browder operator is a Fredholm operator with finite ascent and descent and the Browder spectrum of an operator  $T \in L(X)$  is defined by  $\sigma_{\text{B}}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a Browder operator}\}$ .

In the case of an operator  $T$  acting on a Hilbert space, having generalized Riesz resolvent, we prove that  $T$  has a generalized West decomposition  $T = K + Q + D$ , where  $K$  is a compact operator,  $Q$  is a quasinilpotent operator such that the commutator  $[K, Q]$  is quasinilpotent and  $D$  is a diagonal operator whose spectrum is the Browder spectrum of  $T$ .

## 2 Characterizations of meromorphic operators

We begin this section, by giving some characterizations of bounded linear meromorphic operators in terms of operators of topological uniform descent.

DEFINITION 2.1. [14] Let  $T \in L(X)$  and let  $d \in \mathbf{N}$ . Then  $T$  has a uniform descent for  $n \geq d$  if  $R(T) + N(T^n) = R(T) + N(T^d)$  for all  $n \geq d$ . If in addition  $R(T) + N(T^d)$  is closed then  $T$  is said to have a topological uniform descent for  $n \geq d$ .

DEFINITION 2.2. Let  $T \in L(X)$ ,  $n \in \mathbf{N}$  and let  $c_n(T) = \dim R(T^n)/R(T^{n+1})$  and  $c'_n(T) = \dim N(T^{n+1})/N(T^n)$ .

- The descent of  $T$  is defined by  $\delta(T) = \inf\{n : c_n(T) = 0\} = \inf\{n : R(T^n) = R(T^{n+1})\}$
- Then the ascent  $a(T)$  of  $T$  is defined by  $a(T) = \inf\{n : c'_n(T) = 0\} = \inf\{n : N(T^n) = N(T^{n+1})\}$

THEOREM 2.3. Let  $\Gamma$  be a nonempty connected subset of  $\mathbf{C}$  such that  $T - \lambda I$  is an operator of topological uniform descent for all  $\lambda \in \Gamma$ . If there is  $\alpha \in \Gamma$  such that  $T - \alpha I$  is Drazin invertible, then every point of  $\sigma(T) \cap \Gamma$  is a pole of the resolvent of  $T$  and  $\sigma(T) \cap \Gamma$  is a countable discrete set.

*Proof.* Recall that  $T$  is Drazin invertible if it has a finite ascent and descent. Since  $T - \alpha I$  is Drazin invertible, for  $n$  large enough we have  $c_n(T - \alpha I) = c'_n(T - \alpha I) = 0$ . Let  $A = \{\mu \in \Gamma \mid T - \mu I \text{ is Drazin invertible}\}$ . Then  $\alpha \in A$  and  $A \neq \emptyset$ . If  $\lambda \in A$ , since  $T - \lambda I$  is Drazin invertible, then there is an open neighborhood  $B(\lambda, \epsilon)$  such that  $B(\lambda, \epsilon) \setminus \{\lambda\} \subset \rho(T)$ , where  $\rho(T)$  is the resolvent set of  $T$ . Therefore  $B(\lambda, \epsilon) \cap \Gamma \subset A$ , and  $A$  is open in  $\Gamma$ . Now let  $\lambda \in \overline{A} \cap \Gamma$ , where  $\overline{A}$  is the closure of  $A$ . In particular  $T - \lambda I$  is an operator of topological uniform descent. From [14, Theorem 4.7], there is an  $\epsilon > 0$  such that if  $|\lambda - \mu| < \epsilon$ , then for  $n$  large enough, we have  $c_n(T - \mu I) = c_n(T - \lambda I)$ ,  $c'_n(T - \mu I) = c'_n(T - \lambda I)$ . Since  $\lambda \in \overline{A}$ , then  $B(\lambda, \epsilon) \cap A \neq \emptyset$ . So there is  $\mu \in B(\lambda, \epsilon) \cap A$ . Hence  $c_n(T - \lambda I) = c'_n(T - \lambda I) = 0$ , and so  $\lambda \in A$ . Therefore  $A$  is closed in  $\Gamma$ . Since  $\Gamma$  is connected, then  $A = \Gamma$ . Moreover if  $\lambda \in \sigma(T) \cap \Gamma$ , then  $T - \lambda I$  is Drazin invertible and so  $\lambda$  is a pole of the resolvent of  $T$ . Therefore it is an isolated point of the spectrum  $\sigma(T)$  of  $T$ . Since  $\sigma(T)$  is a compact set, then  $\sigma(T) \cap \Gamma$  is a discrete set.

COROLLARY 2.4. Let  $\Gamma$  be a nonempty connected subset of  $\mathbf{C}$  such that the descent  $\delta(T - \lambda I)$  is finite for all  $\lambda \in \Gamma$ . If there is  $\alpha \in \Gamma$  such that  $T - \alpha I$  is Drazin invertible, then every point of  $\sigma(T) \cap \Gamma$  is a pole of  $T$  and  $\sigma(T) \cap \Gamma$  is a countable discrete set. In particular if  $\Gamma = \mathbf{C}$ , then  $\sigma(T)$  is a countable discrete set of poles of the resolvent of  $T$ .

*Proof.* It's easily seen that in this case  $T - \lambda I$  is an operator of topological uniform descent for all  $\lambda \in \Gamma$ .

Now we extend Theorem 1.1, by including the characterization of meromorphic operators in terms of operators of topological uniform descent.

**THEOREM 2.5.** *Let  $T \in L(X)$ . Then the following conditions are equivalent:*

1.  $T$  is a meromorphic operator
2.  $\sigma_{TUD}(T) \subset \{0\}$ .
3.  $\sigma_{BF}(T) \subset \{0\}$ .

*Proof.* 1)  $\Rightarrow$  2) If  $T$  is a meromorphic operator, then  $T - \lambda I$  is Drazin invertible for each  $\lambda \neq 0$ . In particular  $T - \lambda I$  is an operator of topological uniform descent.

2)  $\Rightarrow$  3) Suppose now that  $\sigma_{TUD}(T) \subset \{0\}$ . Then, for all  $\lambda \neq 0$ ,  $T - \lambda I$  is an operator of topological uniform descent. Let  $\Gamma = \mathbf{C} \setminus \{0\}$ , then  $\Gamma$  is a connected set such that  $T - \lambda I$  is an operator of topological uniform descent for all  $\alpha \in \Gamma$ . Since the spectrum of  $T$  is bounded, then there is  $\lambda \in \Gamma$  such that  $T - \lambda I$  is invertible, and so  $T - \lambda I$  is Drazin invertible. From Theorem 2.5 it follows that every point of  $\Gamma$  is a pole of  $T$ . In particular, for all  $\lambda \neq 0$ ,  $T - \lambda I$  is a B-Fredholm operator and so  $\sigma_{BF}(T) \subset \{0\}$ .

3)  $\Rightarrow$  1) Assume now that  $\sigma_{BF}(T) \subset \{0\}$ , then from Theorem 1.1,  $T$  is a Meromorphic operator.

As shown by the following example, there exists operators which are of topological uniform descent but are not B-Fredholm operators. Thus the previous characterization is a refinement of the characterization of bounded meromorphic operators in terms of B-Fredholm operators.

**EXAMPLE 2.6.** Let  $H$  be a Hilbert space with an orthonormal basis  $\{e_{ij}\}_{i,j=1}^{\infty}$  and let the operator  $T$  defined by :

$$Te_{i,j} = \begin{cases} 0 & \text{if } j = 1, \\ \frac{1}{i}e_{i,1}, & \text{if } j = 2 \\ e_{i,j-1}, & \text{otherwise} \end{cases}$$

It is easily seen that  $R(T) = R(T^2)$  and  $R(T)$  is not closed. Hence  $R(T^n)$  is not closed for all  $n \geq 1$ , and so  $T$  is not a B-Fredholm operator. However, since  $R(T) = R(T^2)$ , then  $T$  is an operator of uniform descent for  $n \geq 1$  and  $N(T) + R(T) = X$ . Hence  $N(T) + R(T)$  is closed. From [14, Theorem 3.2] , it follows that  $T$  is an operator of topological uniform descent for  $n \geq 1$ .

Using Theorem 2.5, we prove in an easy way, as corollaries, some earlier results on meromorphic operators obtained in [9] and [20].

**COROLLARY 2.7.** [9, Theorem 5] *Let  $T \in L(X)$  be a meromorphic operator, and  $f$  an analytic function in a neighborhood  $\sigma(T)$  of  $T$ . If  $f(0) = 0$ , then  $f(T)$  is a meromorphic operator.*

*Proof.* Let  $\sigma_{TUD}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not an operator of topological uniform descent}\}$ . From [3, Theorem 4.3], we have  $\sigma_{TUD}(f(T)) = f(\sigma_{TUD}(T))$ . As  $f(0) = 0$  and  $\sigma_{TUD}(T) \subset \{0\}$ , then  $\sigma_{TUD}(f(T)) \subset \{0\}$ . Hence  $f(T)$  is a meromorphic operator.

In [20], the author has defined semi-finite operators. A bounded linear operator  $T \in L(X)$  is called a semi-finite operator if  $\delta(T)$  is finite or, its ascent  $a(T) = p$  is finite and  $R(T^{p+1})$  is closed.

**COROLLARY 2.8.** [20, Theorem 4.2] *Let  $T \in L(X)$ . Then  $T$  is a meromorphic operator if and only if  $T - \lambda I$  is a semi-finite operator for all  $\lambda \neq 0$ .*

*Proof.* Let  $T \in L(X)$ . It's easily seen that if  $T$  is semi-finite, then  $T$  is an operator of topological uniform descent.

In [9], the subclass  $\mathfrak{F}$  of  $\mathfrak{M}$ , consisting of operators  $T \in L(X)$ , whose spectrum  $\sigma(T)$  is a finite set of poles of the resolvent  $R_\lambda(T)$ , had been considered. It is known that  $T \in \mathfrak{F}$  if and only if  $R_\lambda(T)$  is a rational function [25, p. 336].

**THEOREM 2.9.**  $\mathfrak{F} = \{T \in L(X) \mid \sigma_{TUD}(T) = \emptyset\} = \{T \in L(X) \mid \sigma_{BF}(T) = \emptyset\}$

*Proof.* If  $T \in \mathfrak{F}$ , then  $\sigma(T)$  is a finite set of poles of the resolvent  $R_\lambda(T)$  of  $T$ . Moreover, if  $\lambda$  is a pole of  $T$ , then from [5, Theorem 2.3],  $T - \lambda I$  is a B-Fredholm operator. Hence from [3], it is an operator of topological uniform descent. So  $T - \lambda I$  is an operator of topological uniform descent for all  $\lambda \in \mathbb{C}$ .

Conversely if  $T - \lambda I$  is an operator of topological uniform descent for all  $\lambda \in \mathbb{C}$ , then from Theorem 2.3, with  $\Gamma = \mathbb{C}$ , it follows that every point of  $\sigma(T)$  is a pole of the resolvent of  $T$ . As  $\sigma(T)$  is compact, then it is a finite set of poles of  $T$ .

The second set equality can be proved in exactly the same way.

**REMARK 2.10.** In [26, Corollary 2.10], the authors have obtained the same result as Theorem 2.9, however our method of proof is rather different from their proof.

As a consequence, we obtain the following characterization of the class  $\mathfrak{F}$  proved in [20].

**COROLLARY 2.11.** [20, Theorem 4.3]  $\mathfrak{F} = \{T \in L(X) \mid T - \lambda I \text{ is semi-finite for all } \lambda \in \mathbb{C}\}$ .

Recently Zivkovic-Zlatanovic and al. in [27] and Duggal and al. in [12], studied respectively polynomially Riesz operators and polynomially meromorphic operators. In the following next two theorems, we extend some of their results. We begin by giving the following useful independent result.

PROPOSITION 2.12. *Let  $T \in L(X)$  be such that  $\sigma_{BF}(T)$  is a finite set. Then  $\sigma_D(T) = \sigma_{BF}(T)$  and  $\sigma(T)$  is a discrete set of poles of the resolvent of  $T$ , with its eventual accumulation points contained in  $\sigma_{BF}(T)$ .*

*Proof.* Let  $\sigma_{BF}(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  and let  $P(X)$  be the polynomial defined by  $P(X) = (X - \lambda_1 I)(X - \lambda_2 I) \dots (X - \lambda_n I)$ . Since from [2, Theorem 3.4], the B-Fredholm spectrum satisfies the spectral mapping theorem, we have  $\sigma_{BF}(P(T)) = \{0\}$ . It follows from Theorem 2.5 that  $P(T)$  is a meromorphic operator. So  $\sigma_D(P(T)) \subset \{0\}$ . From [4, Corollary 2.4], we know that  $\sigma_D(T)$  satisfies also the spectral mapping theorem. So  $\sigma_D(T) \subset \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ . As we have always  $\sigma_{BF}(T) \subset \sigma_D(T)$ , then  $\sigma_{BF}(T) = \sigma_D(T)$ . As  $\sigma(T)$  is compact and each pole of the resolvent of  $T$  is an isolated point of the spectrum, then  $\sigma(T) \setminus \sigma_D(T)$  is a discrete set. Therefore  $\sigma(T) = (\sigma(T) \setminus \sigma_D(T)) \cup \sigma_D(T)$  is also a discrete set, with the only possible accumulations points contained in  $\sigma_{BF}(T)$ .

If  $\sigma_{BF}(T)$  is the empty set, then  $\sigma(T)$  is a finite set of poles of the resolvent of  $T$ .

REMARK 2.13. Under the hypothesis of Proposition 2.12, the B-Fredholm spectrum of  $T$  is exactly the set of essential singularities of the resolvent of  $T$ .

Moreover, in this case, if  $\sigma_{BF}(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ , then  $\sigma(T)$  could be partitioned as a finite disjoint union of spectral sets,  $S_1, S_2, \dots, S_n$ . Each set  $S_i, 1 \leq i \leq n$ , is associated to the element  $\lambda_i$  of the B-Fredholm  $\sigma_{BF}(T)$  of  $T$ . The set  $S_i$  could be reduced to the singleton  $\{\lambda_i\}$ , or a finite set containing  $\{\lambda_i\}$  and a finite numbers of poles of the resolvent of  $T$  if  $\lambda_i$  is not an accumulation point of  $\sigma(T)$ , or  $S_i$  is an infinite set containing  $\{\lambda_i\}$  and an infinite sequence of the poles of the resolvent of  $T$  which converges to  $\lambda_i$ .

If  $\sigma_{BF}(T)$  is the empty set, then we define  $S_1 = \sigma(T)$  as the unique spectral set to consider.

THEOREM 2.14. *Let  $T, F \in L(X)$  be two commuting operators such that there exists an integer  $n \geq 1$  with  $F^n$  being a finite rank operator. Then the following conditions are equivalent.*

1. *There exists a non zero polynomial  $P$  such that  $P(T + F)$  is meromorphic.*
2. *There exists an analytic function  $f$  in a neighborhood of the spectrum  $\sigma(T)$  of  $T$ , which is not constant on any connected component of  $\sigma(T)$  such that  $f(T + F)$  is meromorphic.*
3.  *$\sigma_{BF}(T)$  is a finite set, eventually empty.*
4. *There exists a finite set of closed  $T$ -invariant subspaces  $(X_i)_{1 \leq i \leq n}$ , such that  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  and  $T_i = T|_{X_i}$  is a translate of a meromorphic operator for all  $i, 1 \leq i \leq n$ .*



*Proof.* 1)  $\Rightarrow$  2) is trivial

2)  $\Rightarrow$  3) Assume that exists an analytic function  $f$  in a neighborhood of the spectrum  $\sigma(T)$  of  $T$ , which is not constant on any connected component of  $\sigma(T)$ , such that  $f(T+F)$  is meromorphic. Then from Theorem 2.5,  $\sigma_{BF}(f(T+F)) \subset \{0\}$ . From [2, Theorem 3.4], we know that  $\sigma_{BF}(f(T+F)) = f(\sigma_{BF}(T+F))$ . Hence we have  $\sigma_{BF}(T+F) \subset Z_f$ , where  $Z_f$  is the set of the zeros of  $f$  in  $\sigma(T+F)$ . Since  $\sigma(T+F)$  is compact, then  $Z_f$  is a finite set and  $\sigma_{BF}(T+F)$  is also finite. Since  $F$  commutes with  $T$  and  $F^n$  is a finite rank operator, then from [26, Corollary 2.3], we have  $\sigma_{BF}(T+F) = \sigma_{BF}(T)$ . Therefore  $\sigma_{BF}(T)$  is also a finite set, eventually empty.

3)  $\Rightarrow$  4) Assume that  $\sigma_{BF}(T) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$  is a finite set. From Corollary 2.12, we know that  $\sigma(T)$  is a discrete set, whose accumulation points are contained in  $\sigma_{BF}(T)$  and from Remark 2.13,  $\sigma(T)$  could be partitioned as the finite disjoint union of spectral sets,  $S_1, S_2, \dots, S_n$ . Let  $P_i$  be the idempotent associated to the spectral set  $S_i$ , and  $X_i = P_i(X)$ ,  $1 \leq i \leq n$ . Then it is easily seen that  $(X_i)_{1 \leq i \leq n}$  satisfies the requested conditions.

If  $\sigma_{BF}(T) = \emptyset$ , then the resolvent of  $T$ , has no essential singularity in  $\sigma(T)$ . Then  $\sigma(T)$  is a finite set of poles of the resolvent, and similar arguments holds as in the previous case.

4)  $\Rightarrow$  1) Assume that there exists a finite set of closed  $T$ -invariant subspaces  $(X_i)_{1 \leq i \leq n}$ , such that  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  and  $T_i = T|_{X_i}$  is a translate of a meromorphic operator. Then, there exists  $\lambda_i$ ,  $1 \leq i \leq n$ , such that  $T_i - \lambda_i I$  is a meromorphic operator. Let  $P(Z) = (Z - \lambda_1 I_1) \dots (Z - \lambda_n I_n)$ . Then we have  $\sigma_D(P(T+F)) = P(\sigma_D(T+F)) = P(\sigma_D(T)) \subset \{0\}$ . Hence  $P(T+F)$  is a meromorphic operator.

Taking  $F = 0$ , we recover [12, Theorem 4.2] and [17, Theorem 3]. We observe that under the hypothesis of Theorem 2.14, the set  $Q\sigma(T)$  defined in [17, page 85] is exactly the B-Fredholm spectrum  $\sigma_{BF}(T)$  of  $T$ .

From [21, Corollary 2], if  $R$  is a Riesz operator commuting with  $T$ , then  $\sigma_B(T+R) = \sigma_B(T)$ . Following the same steps as in the previous theorem, we obtain the following perturbation result for polynomially Riesz operators, which we give without proof.

**THEOREM 2.15.** *Let  $T \in L(X)$  and let  $R$  be a Riesz operator commuting with  $T$ . Then the following condition are equivalent*

1. *There exists an non zero polynomial  $P$  such that  $P(T+R)$  is a Riesz operator.*
2. *There exists an analytic function  $f$  in a neighborhood of the spectrum, which is not constant on any connected component of the spectrum  $\sigma(T)$  of  $T$  such that  $f(T+R)$  is a Riesz operator.*
3. *The Browder spectrum  $\sigma_B(T)$  is a finite set, eventually empty.*



4. There exists a finite set of closed  $T$ -invariant subspaces  $(X_i)_{1 \leq i \leq n}$ , such that  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  and  $T_i = T|_{X_i}$ , is a translate of a Riesz operator, for all  $i, 1 \leq i \leq n$ .

Taking  $R = 0$ , we obtain stronger versions of [27, Theorem 2.13] and [17, Theorem 1] for non locally constant functions on connected component of the spectrum of  $T$ .

Now, we consider the class  $\mathfrak{M}(0, \infty)$  of unbounded meromorphic operators defined in [9]. It contains the operators  $T$  such that  $\lambda = 0$  and  $\lambda = \infty$ , are the only allowable points of accumulation of  $\sigma(T)$  and every non null isolated point of  $\sigma(T)$  is a pole of the resolvent  $R_{\mu(T)}$  of  $T$ . We prove that an “unbounded version” of Theorem 1.1 characterizes also this class of operators.

**THEOREM 2.16.** *Let  $T$  be a closed operator, with non-empty resolvent set. Then  $T$  belongs to the class  $\mathfrak{M}(0, \infty)$  if and only if*

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a B-Fredholm operator}\} \subset \{0\}$$

*Proof.* If  $\lambda$  is a pole of the resolvent of  $T$ , then from [6, Theorem 2.4],  $T - \lambda I$  is a B-Fredholm operator. Hence, if  $T$  belongs to the class  $\mathfrak{M}(0, \infty)$ , then  $\sigma_{BF}(T) \subset \{0\}$ .

Conversely, assume that  $\sigma_{BF}(T) \subset \{0\}$ . As  $\rho(T)$  is non-empty, there exists  $\lambda_0$  such that  $T - \lambda_0 I$  is invertible. Then from [7, Theorem 3.6], we know that  $\sigma_{BF}((T - \lambda_0 I)^{-1}) \setminus \{0\} = \{\lambda^{-1} \mid \lambda \in \sigma_{BF}(T - \lambda_0 I)\}$ . But  $\sigma_{BF}(T - \lambda_0 I) \subset \{-\lambda_0\}$ . Thus  $\sigma_{BF}((T - \lambda_0 I)^{-1})$  contains at most two points, and its complement set  $\rho_{BF}((T - \lambda_0 I)^{-1})$  is a connected set of B-Fredholm points of the bounded operator  $(T - \lambda_0 I)^{-1}$ . From Theorem 2.3, it follows that each element of  $\rho_{BF}((T - \lambda_0 I)^{-1}) \cap \sigma((T - \lambda_0 I)^{-1})$  is a pole of the resolvent of  $(T - \lambda_0 I)^{-1}$ . Then from [7, Theorem 3.6], each non null element of  $\rho_{BF}((T - \lambda_0 I)) \cap \sigma((T - \lambda_0 I))$  is a pole of the resolvent of  $T - \lambda_0 I$ . Depending on whether  $\lambda_0 = 0$  or not, we can see easily that each non null complex scalar is pole of the resolvent of  $T$ . Hence  $\sigma(T)$  is a discrete set, for which  $\lambda = 0$  and  $\lambda = \infty$ , are the only possible points of accumulation.

### 3 B-discrete spectrum

**DEFINITION 3.1.** Let  $T \in \mathcal{C}(X)$ , with non-empty resolvent set. We will say that  $T$  has a meromorphic resolvent if there exists a scalar  $\lambda$  in the resolvent set  $\rho(T)$  of  $T$  such that  $(T - \lambda I)^{-1}$  is a linear bounded meromorphic operator. Similarly, we will say that  $T$  has a Riesz resolvent if there exists a scalar  $\lambda$  in the resolvent set  $\rho(T)$  of  $T$  such that  $(T - \lambda I)^{-1}$  is a linear bounded Riesz operator.

**REMARK 3.2.** It's easily seen that if  $T$  has meromorphic (resp. Riesz) resolvent, then for all scalar  $\lambda$  in the resolvent set  $\rho(T)$  of  $T$ ,  $(T - \lambda I)^{-1}$  is a linear bounded meromorphic (resp. Riesz) operator.

We begin this section by characterizing operators with purely B-discrete spectrum.

**THEOREM 3.3.** *Let  $T \in \mathcal{C}(X)$ , with a nonempty resolvent set. Then  $T$  has a purely B-discrete spectrum if and only if  $T$  has a meromorphic resolvent.*

*Proof.* Suppose that  $T$  has a purely B-discrete spectrum. So for all  $\lambda \in \mathbb{C}$ ,  $T - \lambda I$  is a B-Fredholm operator of index 0. Since the resolvent set of  $T$  is non-empty, there exists  $\mu \in \mathbb{C}$ , such that  $T - \mu I$  is invertible. From [7, Theorem 3.6],  $(T - \mu I)^{-1} - \lambda I$  is a B-Fredholm operator for all  $\lambda \neq 0$ . Then using Theorem 1.1, we see that  $(T - \mu I)^{-1}$  is a meromorphic operator. Hence  $T$  has a meromorphic resolvent.

Conversely, if  $T$  has a meromorphic resolvent, we can assume without loss of generality, that  $T$  is invertible and that  $T^{-1}$  is a meromorphic operator. If  $\lambda \notin \sigma(T)$ , then  $T - \lambda I$  is invertible and so it is a B-Fredholm operator of index 0. If  $\lambda \in \sigma(T)$ , then  $\lambda \neq 0$ . Since  $T^{-1}$  is a meromorphic operator, then from [7, Theorem 3.6],  $\frac{1}{\lambda}$  is a pole of the resolvent of  $T^{-1}$  and  $\lambda$  is a pole of the resolvent of  $T$ . From [6, Theorem 2.9], it follows that  $T - \lambda I$  is a B-Fredholm operator of index 0. Therefore  $T - \lambda I$  is a B-Weyl operator for all  $\lambda \in \mathbb{C}$  and  $T$  has a purely B-discrete spectrum.

**REMARK 3.4.** An equivalent of Theorem 3.3 for Riesz operators had been proved in [16].

**COROLLARY 3.5.** *Let  $T \in \mathcal{C}(X)$ , with a nonempty resolvent set. If  $T$  has a purely discrete spectrum, then  $T$  has a purely B-discrete spectrum and  $\sigma_{\text{BD}}(T) = \sigma_{\text{D}}(T)$ .*

*Proof.* It follows from [16, Theorem 2], that if  $T$  has purely discrete spectrum, then  $T$  has a Riesz resolvent. So  $T$  has a meromorphic resolvent. Thus  $T$  has a purely B-discrete spectrum. In this case, we have  $\sigma(T) = \sigma_{\text{D}}(T)$ , so  $\sigma_{\text{W}}(T) = \emptyset$ . Hence  $\sigma_{\text{BW}}(T) = \emptyset$  and  $\sigma(T) = \sigma_{\text{BD}}(T) = \sigma_{\text{D}}(T)$ .

As shown by the following example, the converse of the previous corollary is not true in general.

**EXAMPLE 3.6.** [1, Theorem A] Let  $S_B$  be the Schrödinger operator with a constant magnetic field  $B \neq 0$ , in  $\mathbb{R}^2$ ,  $S_B = (\frac{1}{i}\frac{\partial}{\partial x_1} - \frac{Bx_2}{2})^2 + (\frac{1}{i}\frac{\partial}{\partial x_1} + \frac{Bx_1}{2})^2$ . If  $L$  is a self-adjoint extension of  $S_B$  on  $H = L^2(\mathbb{R}^2)$ , then  $\sigma(L) = \{(2k+1) \mid B \mid k \in \mathbb{N}\}$ . Moreover, each eigenvalue of  $L$  has an infinite multiplicity. As  $L$  is self-adjoint, we have  $\sigma(L) = \sigma_{\text{BD}}(L)$ , (See Theorem 3.10). Thus  $L$  has a purely B-discrete spectrum, and its discrete spectrum is empty.

Recall [13] that the Weyl's criterion for the spectrum state that a complex number  $\lambda$  is in the spectrum of a self-adjoint operator  $T$  acting on a Hilbert space  $H$  if and only if there exists a sequence  $(u_n)_n$  in the space  $H$  such that  $\|u_n\| = 1$  and  $\|(T - \lambda)u_n\| \rightarrow 0$ , as  $n \rightarrow \infty$ .

DEFINITION 3.7. [13] Let  $H$  be a Hilbert space, let  $T$  be a self-adjoint operator acting on  $H$  and let  $\lambda \in \mathbb{C}$ . A sequence  $(u_n)_n \subset D(T)$  is called a Weyl sequence for  $T$  at  $\lambda$  if :

- i-  $\forall n \in \mathbb{N}, \|u_n\| = 1$ ,
- ii- The sequence  $(u_n)_n$  converges weakly to 0,
- iii-  $\|(T - \lambda)u_n\| \rightarrow 0$ , as  $n \rightarrow \infty$ .

THEOREM 3.8. [13, Theorem 7.2] Let  $H$  be a Hilbert space, let  $T$  be a self-adjoint operator acting on  $H$  and let  $\lambda \in \mathbb{R}$ . Then  $\lambda$  is in the Weyl spectrum  $\sigma_w(T)$  of  $T$  if and only if there exists a Weyl sequence for  $T$  at  $\lambda$ .

We give now a similar result characterizing the B-Weyl spectrum of a self-adjoint operator.

THEOREM 3.9. Let  $H$  be a Hilbert space, let  $T$  be a self-adjoint operator acting on  $H$ , Let  $\lambda$  be a real scalar. Then  $\lambda$  belongs to the B-Weyl spectrum  $\sigma_{BW}(T)$  of  $T$  if and only if there exists a Weyl sequence for  $T$  at  $\lambda$  and  $\lambda$  is an accumulation point of the spectrum  $\sigma(T)$  of  $T$ .

*Proof.* Let  $\lambda$  be a real scalar such that  $\lambda$  belongs to the B-Weyl spectrum  $\sigma_{BW}(T)$  of  $T$ . As  $\sigma_{BW}(T) \subset \sigma_w(T)$ , then from Theorem 3.8, there exists a Weyl sequence for  $T$  at  $\lambda$ . If  $\lambda$  is isolated in the spectrum  $\sigma(T)$  of  $T$ , then from [19, Theorem 4.4]  $R(T - \lambda I)$  is closed. Hence  $H = N(T - \lambda I) \oplus R(T - \lambda I)$ , and the restriction of the operator  $T - \lambda I : R(T - \lambda I) \rightarrow R(T - \lambda I)$  is invertible. In particular, it is a Fredholm operator of index 0. As  $T$  is self-adjoint, then the resolvent set of  $T - \lambda I$  is non-empty. From [6, Theorem 2.4], it follows that  $T - \lambda I$  is a B-Fredholm operator of index 0. This contradiction shows that  $\lambda$  is not isolated in the spectrum  $\sigma(T)$  of  $T$ .

Conversely suppose that there exists a Weyl sequence for  $T$  at  $\lambda$  and  $\lambda$  is an accumulation point of the spectrum  $\sigma(T)$  of  $T$ . As  $T$  is self-adjoint, the resolvent set of  $T - \lambda I$  is non-empty. If  $T - \lambda I$  is a B-Fredholm operator, then from the proof of [6, Proposition 2.2], there exists an integer  $n \geq 1$  such that  $R((T - \lambda I)^n)$  is closed. Since  $T - \lambda I$  is self-adjoint, then  $N(T - \lambda I) = N((T - \lambda I)^n)$ , and so  $R((T - \lambda I)^n) = N((T - \lambda I)^n)^\perp = N((T - \lambda I))^\perp = \overline{R(T - \lambda I)}$ , the closure of  $R(T - \lambda I)$ . Hence  $\overline{R(T - \lambda I)} \subset R(T - \lambda I)$  and  $R(T - \lambda I)$  is closed. Therefore  $H = N(T - \lambda I) \oplus R(T - \lambda I)$ , and from [20, Corollary 2.2],  $\lambda$  is a pole of the resolvent of  $T$ . Hence it is an isolated point of the spectrum  $\sigma(T)$  of  $T$ , which is a contradiction with the hypothesis on  $\lambda$ . Hence  $\lambda \in \sigma_{BW}(T)$ .

As a consequence of Theorem 3.9, we have the following characterization of the B-discrete spectrum of a self-adjoint operator.

THEOREM 3.10. Let  $H$  be a Hilbert space, let  $T$  be a self-adjoint operator acting on  $H$ . Then the B-discrete spectrum  $\sigma_{BD}(T)$  of  $T$  is the set of all isolated points of the spectrum  $\sigma(T)$  of  $T$  and  $\sigma(T) = \sigma_{BW}(T) \sqcup \sigma_{BD}(T) = \text{acc}(\sigma(T)) \sqcup \text{iso}(\sigma(T))$ .

*Proof.* Let  $\lambda \in \mathbb{R}$  be an accumulation point of the spectrum  $\sigma(T)$  of  $T$ , then  $R(T - \lambda I)$  is not closed, otherwise  $H = N(T - \lambda I) \oplus R(T - \lambda I)$  and so from [20, Corollary 2.2]  $\lambda$  is a pole of the resolvent of  $T$ . Hence  $\lambda$  is isolated in the spectrum  $\sigma(T)$  of  $T$ , which is a contradiction. As  $R(T - \lambda I)$  is not closed, then  $\lambda \in \sigma_w(T)$ . From Theorem 3.8, there exists a Weyl sequence for  $T$  at  $\lambda$ . As  $\lambda$  is an accumulation point of the spectrum  $\sigma(T)$  of  $T$ , then from Theorem 3.9,  $\lambda \in \sigma_{BW}(T)$  and so  $\lambda \notin \sigma_{BD}(T)$ .

Conversely if  $\lambda \in \mathbb{R}$  is isolated in the spectrum  $\sigma(T)$  of  $T$ , then from Theorem 3.9,  $\lambda \notin \sigma_{BW}(T)$ . Hence  $\lambda \in \sigma_{BD}(T)$ . Therefore  $\sigma_{BW}(T) = \text{acc}(\sigma(T))$ ,  $\sigma_{BD}(T) = \text{iso}(\sigma(T))$  and  $\sigma(T) = \sigma_{BW}(T) \sqcup \sigma_{BD}(T) = \text{acc}(\sigma(T)) \sqcup \text{iso}(\sigma(T))$ .

## 4 Operators with generalized Riesz or Meromorphic Resolvent

In the case of an operator  $T$  having a compact resolvent, it is well known that the existence of a complex number  $\lambda \in \rho(T)$  such that  $(T - \lambda I)^{-1}$  is compact, implies that this is the case of all complex numbers in the resolvent set  $\rho(T)$  of  $T$ . This also holds for closed operators having a meromorphic resolvent, as proved in [10, Corollary, page 747]. In this section, we prove similar results for operators having generalized Riesz or generalized meromorphic resolvent.

**DEFINITION 4.1.** Let  $T$  in  $L(X)$ . We will say that  $T$  has a generalized Riesz (respectively a generalized meromorphic) resolvent if there exists a scalar  $\lambda$  in the resolvent set of  $T$  such that  $(T - \lambda I)^{-1}$  is a polynomially Riesz (respectively polynomially meromorphic) operator.

**THEOREM 4.2.** Let  $T$  in  $L(X)$  such that  $T$  has a generalized meromorphic resolvent. Then for all  $\lambda \in \rho(T)$ ,  $(T - \lambda I)^{-1}$  is a polynomially meromorphic operator.

*Proof.* For seek of simplicity, assume that  $T$  is invertible and that  $T^{-1}$  is a polynomially meromorphic operator. Let  $\mu \notin \sigma(T)$  and  $\mu \neq 0$ . Then  $T - \mu I$  is invertible and  $T^{-1} - \frac{1}{\mu}I$  is also invertible. Therefore the function  $f(z) = \frac{z}{1-\mu z}$  is holomorphic in a neighborhood of the spectrum  $\sigma(T^{-1})$  of  $T^{-1}$ . As  $T^{-1}$  is a polynomially meromorphic operator, then from Theorem 2.14,  $\sigma_{BF}(T^{-1})$  is a finite set. So  $\sigma_{BF}(f(T^{-1})) = f(\sigma_{BF}(T^{-1}))$  is also a finite set. Again from Theorem 2.14, the operator  $f(T^{-1})$  is a polynomially meromorphic operator. But  $f(T^{-1}) = T^{-1}(I - \mu T^{-1})^{-1} = [(I - \mu T^{-1})T]^{-1} = (T - \mu I)^{-1}$ , which proves the theorem.

By similar arguments as in the previous theorem, we can prove the following result, which we give without proof.

**THEOREM 4.3.** Let  $T$  in  $L(X)$  such that  $T$  has a generalized Riesz resolvent. Then for all  $\lambda \in \rho(T)$ ,  $(T - \lambda I)^{-1}$  is a polynomially Riesz operator.

In [18], the authors had extended the West decomposition to meromorphic operators and defined the West-Stampfli decomposition as follows.

DEFINITION 4.4. [18, Definition 4.1] Let  $T$  in  $L(X)$ . We will say that  $T$  has a West-Stampfli decomposition if  $T$  can be written as the sum  $T = K + Q$ , where  $K$  is compact and  $\sigma(Q) \subset \text{acc}\sigma(T)$ . We call  $T = M + Q$  a meromorphic West-Stampfli decomposition if  $M$  is a meromorphic operator and  $\sigma(Q) \subset \text{acc}\sigma(T)$ .

We define and study in the following two results, meromorphic generalized West-Stampfli decomposition and generalized West decomposition for operators having generalized meromorphic or generalized Riesz resolvent.

THEOREM 4.5. *Let  $T$  in  $L(X)$  such that  $T$  has a generalized meromorphic resolvent. Then:*

1.  *$T$  is a polynomially meromorphic operator*
2. *The space  $X$  decomposes as a direct sum  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  of closed subspaces  $(X_i)_i, 1 \leq i \leq n$ , such that  $T = \bigoplus_{i=1}^{i=n} (M_i + Q_i + \lambda_i I_i)$ , where  $\lambda_i \in \mathbb{C}$ ,  $I_i : X_i \rightarrow X_i$ , is the identity operator,  $M_i \in L(X_i)$  is meromorphic,  $Q_i \in L(X_i)$  is quasi-nilpotent and the commutators  $[M_i, Q_i], 1 \leq i \leq n$ , are quasi-nilpotent operators.*
3.  *$T$  has a generalized meromorphic West-Stampfli decomposition, that is  $T = M + Q + U$ , where  $M$  is a meromorphic operator,  $Q$  and  $[K, Q]$  are quasi-nilpotent operators and  $U$  is a diagonal operator such that  $\sigma(U) = \sigma_{BF}(T)$ .*

*Proof.* 1-Without loss of generality, we can assume that  $T$  is invertible and  $T^{-1}$  is a polynomially meromorphic operator. Thus  $\sigma_{BF}(T^{-1})$  is a finite set. From [7, Theorem 3.6], it follows that  $\sigma_{BF}(T) = \{\lambda^{-1} \mid \lambda \in \sigma_{BF}(T^{-1})\}$ . Hence  $\sigma_{BF}(T)$  is a finite set and from Theorem 2.14,  $T$  is a polynomially meromorphic operator.

2- Again, from Theorem 2.14, there exists a finite set of closed  $T$ -invariant subspaces  $(X_i)_{1 \leq i \leq n}$ , such that  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  and  $T_i = T|_{X_i}$ , is a translate of a meromorphic operator for all  $i, 1 \leq i \leq n$ . So, for each  $i, 1 \leq i \leq n$ , there exist a scalar  $\lambda_i \in \mathbb{C}$  such that  $T - \lambda_i I_i$ , is a meromorphic operator. From [18, Corollary 4.7], it follows that  $T_i - \lambda_i I_i = M_i + Q_i$ , where  $M_i$  is meromorphic,  $Q_i$  and the commutator  $[M_i, Q_i]$  are quasinilpotent operators. Thus  $T = \bigoplus_{i=1}^{i=n} (M_i + Q_i + \lambda_i I_i)$ .

3- Let  $M = \bigoplus_{i=1}^{i=n} M_i$ ,  $Q = \bigoplus_{i=1}^{i=n} Q_i$  and  $U = \bigoplus_{i=1}^{i=n} \lambda_i I_i$ . Then  $T = M + Q + U$ , the commutator  $[M, Q]$  is quasi-nilpotent and  $U$  is a diagonal operator such that  $\sigma(U) = \sigma_{BF}(T)$ . From the proof of Theorem 2.14, it follows that  $\sigma(U) = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma_{BF}(T)$ . Thus  $T$  has a generalized meromorphic West-Stampfli decomposition.

For operators with generalized Riesz resolvent, acting on a Banach space, the following result extend [18, Theorem 4.13] and extends also a result of [24], where it was assumed that  $\sum_{n=1}^{\infty} n |\nu_n| < \infty$ .

**THEOREM 4.6.** *Let  $T$  in  $L(X)$  be such that  $T$  has a generalized Riesz resolvent. Assume also that  $\sum_{n=1}^{\infty} |\nu_n| < \infty$ , where  $(\nu_n)_n$  are all isolated eigenvalues of  $T$ , each repeated according to its multiplicity. Then:*

1.  $T$  is a polynomially Riesz operator
2. The space  $X$  decomposes as a direct sum  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$ , of closed subspaces  $(X_i)_i, 1 \leq i \leq n$ , such that  $T = \bigoplus_{i=1}^{i=n} (K_i + Q_i + \lambda_i I_i)$ , where  $\lambda_i \in \mathbb{C}, I_i : H_i \rightarrow H_i$  is the identity operator,  $K_i \in L(H_i)$  is compact,  $Q_i \in L(X_i)$  is quasi-nilpotent and the commutators  $[K_i, Q_i], 1 \leq i \leq n$ , are quasi-nilpotent operators.
3.  $T$  has a generalized West decomposition  $T = K + Q + U$ , where  $K$  is a compact operator,  $Q$  and  $[K, Q]$  are quasi-nilpotent operators and  $U$  is a diagonal operator such that  $\sigma(U) = \sigma_B(T)$ .

*Proof.* 1-Without loss of generality, we can assume that  $T$  is invertible and  $T^{-1}$  is a polynomially Riesz operator. Thus  $\sigma_B(T^{-1})$ , is a finite set. From [7, Theorem 3.6], it follows that  $\sigma_B(T) = \{\lambda^{-1} \mid \lambda \in \sigma_{BF}(T^{-1})\}$ . Hence  $\sigma_{BF}(T)$  is a finite set and from Theorem 2.15,  $T$  is a polynomially Riesz operator.

2- Again from Theorem 2.15, there exists a finite set of closed  $T$ -invariant subspaces  $(X_i)_{1 \leq i \leq n}$ , such that  $X = X_1 \oplus X_2 \oplus \dots \oplus X_n$  and  $T_i = T|_{X_i}$ , is a translate of a Riesz operator for all  $i, 1 \leq i \leq n$ . So, for each  $i, 1 \leq i \leq n$ , there exist a scalar  $\lambda_i \in \mathbb{C}$  such that  $T - \lambda_i I_i$ , is a Riesz operator. From [18, Theorem 4.13], it follows that  $T_i - \lambda_i I_i = K_i + Q_i$ , where  $K_i$  is compact,  $Q_i$  is quasi-nilpotent, the commutator  $[K_i, Q_i]$  is a quasinilpotent operator. Thus  $T = \bigoplus_{i=1}^{i=n} (K_i + Q_i + \lambda_i I_i)$ .

3- Let  $K = \bigoplus_{i=1}^{i=n} K_i$ ,  $Q = \bigoplus_{i=1}^{i=n} Q_i$  and  $U = \bigoplus_{i=1}^{i=n} \lambda_i I_i$ . Then  $T = K + Q + U$ , the commutator  $[K, Q]$  is quasi-nilpotent and  $U$  is a diagonal operator such that  $\sigma(U) = \{\lambda_1, \lambda_2, \dots, \lambda_n\} = \sigma_B(T)$ . Then  $T$  has a generalized West decomposition.

When  $T$  is an operator acting on Hilbert space, having a generalized Riesz resolvent, we can use the West decomposition [11, Corollary ] to obtain in the same way as in the previous theorem a generalized West decomposition for  $T$ . We mention that this result had been also proved in [15, Lemma 3], but without proving that the commutators  $[K_i, Q_i], 1 \leq i \leq n$ , are quasi-nilpotent.

**THEOREM 4.7.** *Let  $H$  be a Hilbert space and let  $T$  in  $L(H)$  has a generalized Riesz resolvent. Then:*

1.  $T$  is a polynomially Riesz operator.
2. The space  $H$  decomposes as a direct sum  $H = H_1 \oplus H_2 \oplus \dots \oplus H_n$ , of closed subspaces  $(H_i)_i, 1 \leq i \leq n$ , such that  $T = \bigoplus_{i=1}^n (K_i + Q_i + \lambda_i I_i)$ , where  $\lambda_i \in \mathbb{C}, I_i : H_i \rightarrow H_i$ , is the identity operator,  $K_i \in L(H_i)$  is compact,  $Q_i \in L(H_i)$  is quasi-nilpotent and the commutators  $[K_i, Q_i], 1 \leq i \leq n$ , are quasi-nilpotent operators.
3.  $T = K + Q + U$ , where  $K$  is a compact operator,  $Q$  and  $[K, Q]$  are quasi-nilpotent operators, and  $U$  is a diagonal operator such that  $\sigma(U) = \sigma_B(T)$ .

*Proof.* Instead of [18, Theorem 4.13], we use [11, Corollary ], which says that a Riesz operator is the sum of a compact operator and a quasi-nilpotent one with their commutator being quasi-nilpotent, and we follow the same steps as in Theorem 4.6.

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